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LETTER TO THE EDITOR

Existence of the ferromagnetic phase in a random-bond Ising model on the square lattice

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Abstract. We extend Griffiths' arguments to show that there exists the thermodynamic limit in a quenched random-bond Ising model with competing interactions on the square lattice. We extend Peierls' arguments to give 0.97394 for an upper bound of the critical concentration of the ferromagnetic phase for the system in which the exchange integrals take on $J>0$ and $-J$ with respective probabilities p and $1-p$.

The random-bond Ising model with exchange integrals $J>0$ and $-J$ has arrested the attention of many authors on the problem of the existence of the spin-glass phase defined by Edwards and Anderson (1975). Apart from results obtained by approximate calculations, the exact nature of the system is not much clarified (e.g. see Nishimori 1980, 1981a, b, Morita and Horiguchi 1980, Horiguchi 1981a, b). The disappearance of the long-range order parameter is necessary for the existence of the spin-glass phase. The present authors showed that there is no spontaneous long-range order in a finite interval of concentration p of the ferromagnetic bonds (Horiguchi and Morita 1981, 1982a, b). Recently, Nishimori (1981b) applied Peierls' arguments (Peierls 1936) modified by Griffiths (1964a, 1972) and showed that there is a region on the p–T plane in which the ferromagnetic phase surely occurs; T is the temperature. However, his results only give a trivial upper bound $p=1$ of the critical concentration of ferromagnetic bonds for the occurrence of the ferromagnetic phase at $T=0$. We believe that there should exist a critical concentration p_c less than 1. Our aim in the present Letter is to find a non-trivial upper bound of the critical concentration of the ferromagnetic bonds for the occurrence of the ferromagnetic phase by extending Peierls' and Griffiths' arguments.

We consider a random-bond Ising model on a square lattice Λ. The total number of lattice sites is denoted by $|\Lambda|$. The Hamiltonian of the system is assumed to be given by

$$H=-\sum_{\substack{(ij)\\ i,j\in\Lambda}} J_{ij}s_is_j-h\sum_{i\in\Lambda} s_i \qquad (1)$$

where s_i is the spin variable for site i and takes on the values ±1 and h is the external field. The first summation on the right-hand side is taken over all nearest-neighbour pairs of sites. J_{ij} are the exchange integrals which are quenched random variables and whose probability distribution is denoted by $P(J_{ij})$ and assumed to be independent of those of the J_{kl} for other bonds (kl). We denote the configurational average of a function $Q\{J_{ij}\}$ of the set $\{J_{ij}\}$ by the angular brackets with a subscript c:

$$\langle Q\{J_{ij}\}\rangle_{c}=\int Q\{J_{ij}\}\prod_{(ij)}[P(J_{ij})\,\mathrm{d}J_{ij}]. \tag{2}$$

We assume that $\langle|J_{ij}|\rangle_c$ is finite.

The free energy associated with the Hamiltonian H is defined by

$$F(H)=-\beta^{-1}\ln\sum_{\{s_i\}}\mathrm{e}^{-\beta H} \tag{3}$$

where $\beta=1/k_{\mathrm{B}}T$ as usual. We divide H into two parts, H_0 and H_1: $H=H_0+H_1$. Since our system is classical, we easily obtain the inequality

$$\langle\langle H_1\rangle\rangle_c\leqslant\langle F(H)\rangle_c-\langle F(H_0)\rangle_c\leqslant\langle\langle H_1\rangle_0\rangle_c. \tag{4}$$

Here the angular brackets $\langle\mathscr{O}\rangle$ denote the canonical average of a quantity $\mathscr{O}$ for the system with the Hamiltonian H

$$\langle\mathscr{O}\rangle=\sum_{\{s_i\}}\mathrm{e}^{-\beta H}\mathscr{O}\Big/\sum_{\{s_i\}}\mathrm{e}^{-\beta H} \tag{5}$$

and $\langle\mathscr{O}\rangle_0$ for the system with the Hamiltonian H_0. From equation (4) we have

$$|\langle F(H)\rangle_c-\langle F(H_0)\rangle_c|\leqslant\langle\max_{\{s_i\}}|H_1|\rangle_c. \tag{6}$$

In order to prove the existence of the thermodynamic limit following Griffiths (1964b, 1972) and Griffiths and Lebowitz (1968), we consider a system on a rectangle with sides of MP and NQ lattice sites. We assume that $MPNQ$ lattice sites are composed of PQ smaller rectangles, each with sides of M and N lattice sites. The Hamiltonian is denoted by $H(MP, NQ)$ and we rewrite it as

$$H(MP,NQ)=\sum_{l=1}^{PQ}H_l(M,N)+H'. \tag{7}$$

H' is the part of the Hamiltonian expressing the exchange interactions between the nearest-neighbour sites i and j in the original lattice where the site j belongs to a rectangle adjacent to the one to which the site i belongs. We introduce the notations

$$f(MP,NQ)=\langle F(H(MP,NQ))\rangle_c/MPNQ \tag{8}$$

and

$$f(M,N)=\langle F(H_l(M,N))\rangle_c/MN \tag{9}$$

and we arrive immediately from equation (6) at

$$|f(MP,NQ)-f(M,N)|\leqslant(M^{-1}+N^{-1})\langle|J_{ij}|\rangle_c. \tag{10}$$

We note that equation (10) holds for arbitrary integers M, N, P and Q. When we choose $K=4\langle|J_{ij}|\rangle_c/\varepsilon$ for an arbitrary positive ε, we have from equation (10) for arbitrary integers M_1, N_1, M_2 and N_2 greater than K:

$$|f(M_1,N_1)-f(M_2,N_2)|<\varepsilon. \tag{11}$$

Thus we prove that $f(M, N)$ approaches a limiting value $\bar{f}$ as both M and N increase to infinity. It is obvious from equation (6) that the boundary conditions have negligible effects for a large rectangle because the correction terms become negligible as the surface-to-volume ratio approaches zero in the sense of van Hove (Ruelle 1969).

From equation (4), we have

$$\langle F(H_0+xH_1)\rangle_c \leqslant \langle F(H_0)\rangle_c + x\langle\langle H_1\rangle_0\rangle_c. \tag{12}$$

When we choose H_0 as the exchange energy part and H_1 as the external field part, we have

$$\bar{f}_B(h) \leqslant \bar{f}_B(h=0) - hm_B(h=0) \tag{13}$$

where

$$m_B(h=0) = \frac{1}{|\Lambda|}\sum_{i\in\Lambda} \langle\langle s_i\rangle_{0,B}\rangle_c. \tag{14}$$

Here the suffix B means a boundary condition imposed on the system. We shall consider the boundary condition B_0 that the boundary spins are not coupled with an outer system and the one B_1 that the boundary spins are forced to align upwards.

Since the special boundary condition B_1 results only in a surface contribution of the free energy, we have $\bar{f}_{B_1}(h) = \bar{f}_{B_0}(h)$, and hence we have

$$\bar{f}_{B_0}(h) \leqslant \bar{f}_{B_0}(h=0) - hm_{B_1}(h=0). \tag{15}$$

The spontaneous magnetisation is defined by

$$m_s = -\lim_{h\to 0}\left(\frac{\partial \bar{f}_{B_0}(h)}{\partial h}\right)_T. \tag{16}$$

If we have a positive number α for which $m_{B_1}(h=0) \geqslant \alpha > 0$, then we conclude that m_s cannot be less than α since $\bar{f}_{B_0}(h)$ is a concave, symmetric function of h at any fixed temperature.

We now show that the magnetisation $m_{B_1}(h=0)$ has a positive lower bound α at low temperatures. For $h=0$, equation (1) is written as follows under the boundary condition B_1

$$H = -\sum_{\substack{(ij)\\ i,j\in\Lambda\backslash\bar{\Lambda}}} J_{ij}s_is_j - \sum_{\substack{(ij)\\ i\in\Lambda\backslash\bar{\Lambda},\, j\in\bar{\Lambda}}} J_{ij}s_i \tag{17}$$

where $\bar{\Lambda}$ is the set of sites j which are on the boundary of the lattice Λ. Here we restrict our system to the one that $P(J_{ij})$ is p for $J_{ij}=J>0$ and $1-p$ for $J_{ij}=-J$ and $\frac{1}{2}\leqslant p\leqslant 1$. Following Griffiths (1964a, 1972), we draw lines bisecting pairs of sites i and j if s_i is $+1$ on one site i and s_j is -1 on the other site j. These form closed polygons each of which encloses at most $(\frac{1}{4}b)^2$ lattice sites if its perimeter is b. Thus the magnetisation is expressed as

$$m_{B_1}(h=0) = 1 - 2\langle\langle N_-\rangle\rangle_c/|\Lambda| \tag{18}$$

and

$$N_- \leqslant \sum_{b=4,6,\ldots} (\tfrac{1}{4}b)^2 \sum_{j=1}^{\nu(b)} X_b^{(j)} \tag{19}$$

where $X_b^{(j)}$ is 1 if the jth border of length b occurs in a spin configuration and 0 otherwise. Here N_- is the total number of sites with down spins in the system and $\nu(b)$ is the number of different polygons of perimeter b in the system. Taking the thermal average of equation (19) with Hamiltonian (17), we have

$$\langle N_-\rangle \leq \sum_{b=4,6,\ldots} (\tfrac{1}{4}b)^2 \sum_{j=1}^{\nu(b)} \langle X_b^{(j)}\rangle. \tag{20}$$

The thermal average of $X_b^{(j)}$ is expressed as

$$\langle X_b^{(j)}\rangle = \sum_{\{s_i\}}{}' \, \mathrm{e}^{-\beta H} \Big/ \sum_{\{s_i\}} \mathrm{e}^{-\beta H}. \tag{21}$$

The sum in the denominator is taken over all the configurations of spin states satisfying the boundary condition, but the sum in the numerator only over all the configurations in which the jth border of length b appears. In order to find an upper bound of $\langle X_b^{(j)}\rangle$, we restrict the sum in the denominator to some special configurations. For the case in which the sum of J_{ij} over the nearest-neighbour pairs of sites (ij) across the border is zero or negative, we consider only the spin configurations that appear in the numerator. Thus we have an upper bound

$$\langle X_b^{(j)}\rangle \leq 1 \tag{22}$$

when $\Sigma_{(ij)\in\mathscr{B}(b)} J_{ij} \leq 0$ where $\mathscr{B}(b)$ is the set of nearest-neighbour pairs of sites (ij) which are situated across the border of length b. For the case in which the sum of J_{ij} is positive, we consider only the spin configurations which are generated from the spin configurations in the numerator by reversing all the spins inside their border. Thus we have an upper bound

$$\langle X_b^{(j)}\rangle \leq \exp\Big(-2\beta \sum_{(ij)\in\mathscr{B}(b)} J_{ij}\Big) \tag{23}$$

when $\Sigma_{(ij)\in\mathscr{B}(b)} J_{ij} > 0$.

Now we take the configurational average of equation (21) by using equations (22) and (23) and find an upper bound

$$\begin{aligned}\langle\langle X_{2n}^{(j)}\rangle\rangle_c &\leq \sum_{l=0}^{n} \binom{2n}{l} p^l (1-p)^{2n-l} + \sum_{l=n+1}^{2n} \binom{2n}{l} p^l (1-p)^{2n-l} \lambda^{2l-2n} \\ &= \sum_{l=0}^{n} \binom{2n}{l} p^l (1-p)^{2n-l} + \sum_{l=0}^{n-1} \binom{2n}{l} (p\lambda)^{2n-l} \left(\frac{1-p}{\lambda}\right)^l \end{aligned} \tag{24}$$

where $\lambda = \exp(-2\beta J)$. We shall use for $\nu(b)$ the value estimated by Griffiths and Lebowitz (1968):

$$\nu(b) \leq 2|\Lambda| 3^{b-2}/b. \tag{25}$$

This value is half of their value since we need polygons without direction. Equation (20) is now overestimated by

$$\frac{1}{|\Lambda|}\langle\langle N_-\rangle\rangle_c \leq \frac{1}{36}\sum_{n=2}^{\infty} n 9^n \langle\langle X_{2n}^{(j)}\rangle\rangle_c \tag{26}$$

$$\leq \frac{1}{36}\sum_{n=2}^{\infty} n[9p(1-p)]^n \left[\sum_{l=0}^{n} \binom{2n}{n-l}\left(\frac{1-p}{p}\right)^l + \sum_{l=1}^{n} \binom{2n}{n-l}\left(\frac{p\lambda^2}{1-p}\right)^l\right]. \tag{27}$$

The series in equation (27) converges at $T=0$ in the range of p given by $0.97140 < p \leq$

1. The condition that the right-hand side of equation (27) is equal to $\frac{1}{2}$ is obtained numerically and shown in figure 1 by the solid line bordering the hatched region in the p–T plane. We find 0.97394 for an upper bound of the critical concentration p_c above which the ferromagnetic phase occurs at $T = 0$. In the hatched region, we certainly have spontaneous magnetisation. In figure 1, the vertical broken line shows that there is no spontaneous long-range order to the left of the line 0.70710 down to 0.29289 and the horizontal broken line shows that there is no spontaneous long-range order above it (Horiguchi and Morita 1981).

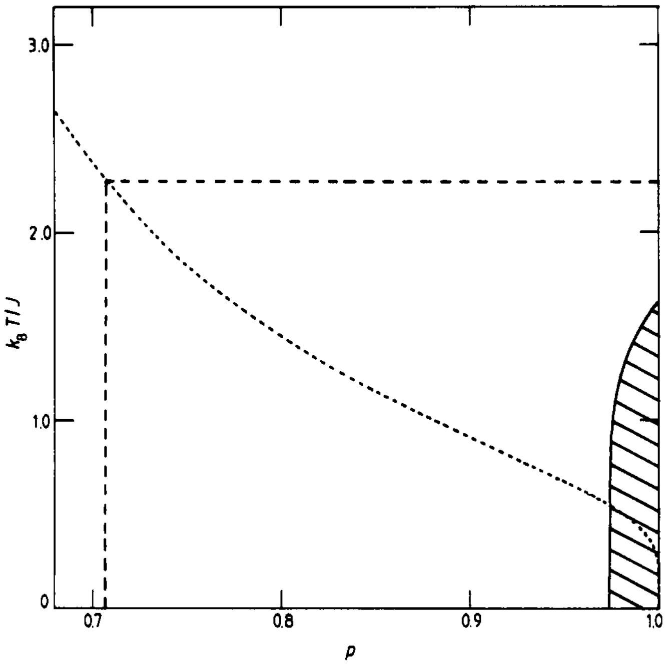

Figure 1. The region hatched in the p–T plane where there certainly exists the spontaneous magnetisation and the region outside the broken line where the spontaneous magnetisation certainly does not exist. The dotted line is Nishimori's line.

Thus in this Letter we could show that there certainly exists the critical concentration for the occurrence of the ferromagnetic phase in the random-bond Ising model with competing interactions $\pm J$. On the other hand, we have already shown that there is no spontaneous long-range order in a range of parameter p as also shown in figure 1. By combining both results, we have ultimately proved that the phase transition occurs in the system as a function of concentration p at sufficiently low temperatures. Our arguments are easily extended to other two- and three-dimensional lattices and also to the occurrence of the antiferromagnetic phase.

An improved upper bound could be obtained if we used both of the configurations used in estimating (22) and (23) in the denominator of (21) and also if we used a better upper bound for $\nu(b)$ obtained in the theory of self-avoiding walks (e.g. see Domb and Hioe 1970).

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